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Bethe equations 'on the wrong side of the equator'

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Abstract. We analyse Baxter's famous $T-Q$ equations for the XXX (XXZ) spin chain and show that, apart from its usual polynomial (trigonometric) solution, which provides the solution of Bethe ansatz equations, there also exists a second solution which should correspond to the Bethe ansatz beyond $N/2$. This second solution of Baxter's equation plays an essential role and together with the first one gives rise to all fusion relations.

1. Associated solutions of Bethe ansatz equations for XXX-spin chains

The equations of the Bethe ansatz in the case of the XXX-spin-$\frac{1}{2}$ chain [1] can be written in the following form (see, e.g., [2]):

$$\left(\frac{\lambda_j + \mathrm{i}/2}{\lambda_j - \mathrm{i}/2}\right)^N = \prod_{k\neq j}^{n} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k - \mathrm{i}} = -\prod_{k=1}^{n} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k - \mathrm{i}} \qquad (j = 1, 2, \ldots, n) \tag{1}$$

where N is the length of the chain (total number of spins) and n is the number of parameters λ_j, which describe the state vector.

The total spin of the eigenstate, described by λ_j, is equal to $N/2 - n$, therefore only states with $n \leqslant N/2$ are meaningful. One can prove, for example, within the framework of the quantum inverse scattering method (QISM) (see, e.g., [3]), that, if $n > N/2$, the corresponding Bethe vector vanishes.

Nevertheless, the solutions of (1) with n beyond the equator $N/2$ do exist and, moreover, their consideration appears to be very useful.

In this section we shall prove the following.

Theorem 1 (Extended Bethe ansatz for the XXX spin chain). *For each solution of (1) with $n \leqslant N/2$ there exists the associated one-parametric solution with $n^* = N - n + 1 > N/2$.*

Proof.

- Let us consider the set $\{\lambda_j\}$, which is the solution of (1), with $n \leqslant N/2$. This set defines the polynomial $Q(\lambda)$†, whose roots are $\{\lambda_j\}$:

$$Q(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j). \tag{2}$$

† In the more general situation of an inhomogeneous XXZ spin chain this polynomial was introduced by Baxter [4].

Equations (1) could be represented in the following form:

$$(\lambda_j - i/2)^N Q(\lambda_j + i) + (\lambda_j + i/2)^N Q(\lambda_j - i) = 0 \qquad (j = 1, 2, \dots, n) \tag{3}$$

from which it follows that the polynomial of degree $N + n$

$$(\lambda - i/2)^N Q(\lambda + i) + (\lambda + i/2)^N Q(\lambda - i) \tag{4}$$

vanishes at the roots of the polynomial $Q(\lambda)$. For the case of the simple roots this statement implies the validity of the Baxter equation for XXX spin chain [5, 6]:

$$(\lambda - i/2)^N Q(\lambda + i) + (\lambda + i/2)^N Q(\lambda - i) = T(\lambda)\, Q(\lambda) \tag{5}$$

where the polynomial $T(\lambda)$ of degree N, is an eigenvalue of the transfer matrix (the trace of the monodromy matrix) for the XXX model.

- Let us divide both sides of (5) on the product $Q(\lambda - i)\, Q(\lambda)\, Q(\lambda + i)$

$$\frac{T(\lambda)}{Q(\lambda + i)\, Q(\lambda - i)} = R(\lambda - i/2) + R(\lambda + i/2) \tag{6}$$

where

$$R(\lambda) = \frac{\lambda^N}{Q(\lambda - i/2)\, Q(\lambda + i/2)}. \tag{7}$$

The rational function $R(\lambda)$ can be presented in the following form:

$$R(\lambda) = \pi(\lambda) + \frac{q_-(\lambda)}{Q(\lambda - i/2)} + \frac{q_+(\lambda)}{Q(\lambda + i/2)} \tag{8}$$

where $\pi(\lambda)$, $q_-(\lambda)$ and $q_+(\lambda)$ are polynomials, whose degrees satisfy

$$\begin{aligned} &\deg \pi(\lambda) = N - 2n \\ &\deg q_-(\lambda) < n \\ &\deg q_+(\lambda) < n. \end{aligned} \tag{9}$$

These inequalities will be used in the following.

Making use of the representation (8) for $R(\lambda)$ let us rewrite equation (6),

$$\begin{aligned} \frac{T(\lambda)}{Q(\lambda + i)\, Q(\lambda - i)} &= \pi(\lambda - i/2) + \pi(\lambda + i/2) \\ &+ \frac{q_-(\lambda - i/2)}{Q(\lambda - i)} + \frac{q_+(\lambda - i/2)}{Q(\lambda)} + \frac{q_-(\lambda + i/2)}{Q(\lambda)} + \frac{q_+(\lambda + i/2)}{Q(\lambda + i)}. \end{aligned} \tag{10}$$

In the right-hand side of (10) there are two terms with the denominator $Q(\lambda)$:

$$\frac{q_+(\lambda - i/2) + q_-(\lambda + i/2)}{Q(\lambda)}$$

which are absent in the left-hand side. The degree of the nominator of this fraction according to (9) is less than the degree of the denominator, therefore the two terms should cancel each other, hence

$$q_+(\lambda) = q(\lambda + i/2) \qquad q_- = -q(\lambda - i/2). \tag{11}$$

- With (11) the representation for $R(\lambda)$ becomes

$$R(\lambda) = \pi(\lambda) + \frac{q(\lambda + i/2)}{Q(\lambda + i/2)} - \frac{q(\lambda - i/2)}{Q(\lambda - i/2)}. \tag{12}$$

The polynomial $\pi(\lambda)$, as any other, may also be presented as the finite difference

$$\pi(\lambda) = \rho(\lambda + i/2) - \rho(\lambda - i/2) \tag{13}$$

where $\rho(\lambda)$ is a polynomial of degree $N - 2n + 1$. The explicit form of $\rho(\lambda)$ one can obtain, for example, via binomial polynomials $\binom{\lambda}{m}$, $m = 0, 1, 2, \dots$.

- Taking into account (13) we arrive at the following equation for our rational function $R(\lambda)$:

$$R(\lambda) \equiv \frac{\lambda^N}{Q(\lambda + \mathrm{i}/2)\, Q(\lambda - \mathrm{i}/2)} = \frac{P(\lambda + \mathrm{i}/2)}{Q(\lambda + \mathrm{i}/2)} - \frac{P(\lambda - \mathrm{i}/2)}{Q(\lambda - \mathrm{i}/2)} \tag{14}$$

where $P(\lambda)$ is the last and most important polynomial of this theorem:

$$P(\lambda) = \rho(\lambda)\, Q(\lambda) + q(\lambda). \tag{15}$$

Counting the degree of $P(\lambda)$ we obtain $\deg P(\lambda) = n^* = N + 1 - n$.

- Now we can get rid of the denominators in (14), and obtain the fundamental equation

$$P(\lambda + \mathrm{i}/2)\, Q(\lambda - \mathrm{i}/2) - P(\lambda - \mathrm{i}/2)\, Q(\lambda + \mathrm{i}/2) = \lambda^N. \tag{16}$$

- This equation is invariant under the substitution $Q \to -P$, therefore the roots of the polynomial $P(\lambda)$, which we denote as $\{\lambda_j^*\}$ provide the solution of Bethe ansatz equations:

$$\left(\frac{\lambda_j^* + \mathrm{i}/2}{\lambda_j^* - \mathrm{i}/2}\right)^N = \prod_{k \neq j}^{n^*} \frac{\lambda_j^* - \lambda_k^* + \mathrm{i}}{\lambda_j^* - \lambda_k^* - \mathrm{i}} \qquad (j = 1, 2, \ldots, n^*) \tag{17}$$

as the roots of $Q(\lambda)$ provide the solution of (1).

- The polynomial $\rho(\lambda)$ in (13) is defined up to the arbitrary constant α. This implies that the polynomial $P(\lambda)$, corresponding to $Q(\lambda)$ is actually the one-parametric family

$$P(\lambda, \alpha) = P(\lambda) + \alpha Q(\lambda) \tag{18}$$

with obvious agreement with (16). □

The theorem we have just proven may be illustrated by the concrete example of the set of polynomials P and Q for the case $N = 4$.

Table 1.

Number	S	$Q(\lambda)$	$(2S+1)\, i\, P(\lambda)$	$T(\lambda)$
1	0	$\lambda^2 + \frac{1}{4}$	$\lambda^3 + \frac{5}{4}\lambda + \alpha(\lambda^2 + \frac{1}{4})$	$2\lambda^4 + 3\lambda^2 - \frac{3}{8}$
2	0	$\lambda^2 - \frac{1}{12}$	$\lambda^3 + \frac{1}{4}\lambda + \alpha(\lambda^2 - \frac{1}{12})$	$2\lambda^4 + 3\lambda^2 + \frac{13}{8}$
3	1	$\lambda - \frac{1}{2}$	$\lambda^4 + \lambda^3 + \lambda^2 + \frac{5}{8}\lambda + \alpha(\lambda - \frac{1}{2})$	$2\lambda^4 + \lambda^2 + 2\lambda + \frac{1}{8}$
4	1	$\lambda + \frac{1}{2}$	$\lambda^4 - \lambda^3 + \lambda^2 - \frac{5}{8}\lambda + \alpha(\lambda + \frac{1}{2})$	$2\lambda^4 + \lambda^2 - 2\lambda + \frac{1}{8}$
5	1	λ	$\lambda^4 - \frac{1}{2}\lambda^2 - \frac{3}{16} + \alpha\lambda$	$2\lambda^4 + \lambda^2 - \frac{7}{8}$
6	2	1	$\lambda^5 + \frac{5}{6}\lambda^3 + \frac{7}{48}\lambda + \alpha$	$2\lambda^4 - 3\lambda^2 + \frac{1}{8}$

A few comments are in order.

- The polynomial $Q(\lambda)$ is normalized in such a way that the coefficient at the highest degree is equal to 1. Comparing the highest degrees in (16), we obtain that the coefficient at the highest degree of $P(\lambda)$ is equal to $1/\mathrm{i}(N - 2n + 1) = 1/\mathrm{i}(2S + 1)$, where S is the spin of Bethe state.
- Note that the existence of a one-parametric solution of Bethe equations 'beyond the equator' implies that these equations are not independent. We shall consider the consequences of this fact in a separate paper.
- Let us return to equation (6). Using the representation (14) for $R(\lambda)$, we obtain the following expression for eigenvalues of the transfer matrix:

$$T(\lambda) = P(\lambda + \mathrm{i})\, Q(\lambda - \mathrm{i}) - P(\lambda - \mathrm{i})\, Q(\lambda + \mathrm{i}). \tag{19}$$

- Combining (16) and (19) we easily obtain the equation:

$$(\lambda - \mathrm{i}/2)^N P(\lambda + \mathrm{i}) + (\lambda + \mathrm{i}/2)^N P(\lambda - \mathrm{i}) = T(\lambda) P(\lambda) \qquad (20)$$

  similar to the Baxter equation (5).
  This means that $P(\lambda)$ may be considered as the second independent solution of (5). The arbitrary linear combination of Q and P is the solution of the Baxter equation as well.
- Finally, note that the polynomials $Q(\lambda)$, $P(\lambda)$, satisfying the relation (16) are similar to the eigenvalues of operators $Q\pm$, which have been constructed in the series of papers by Bazhanov *et al* [7]. In these papers authors considered the field theory analogues of some useful constructions of lattice integrable models. The extension of their $Q\pm$ operators for the six-vertex model† requires an external magnetic field. Unfortunately at the present stage we do not know whether there exists a direct relation of their operators with our $P - Q$ polynomials. The general construction of Krichiver *et al* [9] also uses a special parameter ν which plays the role of the magnetic field.

2. Fusion relations for transfer matrices

As was emphasized in [7, 9], the fundamental equation

$$P(\lambda + \mathrm{i}/2)\, Q(\lambda - \mathrm{i}/2) - P(\lambda - \mathrm{i}/2)\, Q(\lambda + \mathrm{i}/2) = \lambda^N \qquad (21)$$

implies the existence of the class of functional relations known as fusion relations for transfer matrices (see, e.g., [10]).

Now we have shown that the fundamental relation (21) follows from the Bethe ansatz equations, therefore these fusion relations also arise due to the Bethe ansatz.

Let us consider the details of the connection of (21) and fusion relations.

First of all let us define the functions $T_s(\lambda)$ as follows:

$$T_s(\lambda) = P\big(\lambda + \mathrm{i}\big(s + \tfrac{1}{2}\big)\big)\, Q\big(\lambda - \mathrm{i}\big(s + \tfrac{1}{2}\big)\big) - P\big(\lambda - \mathrm{i}\big(s + \tfrac{1}{2}\big)\big)\, Q\big(\lambda + \mathrm{i}\big(s + \tfrac{1}{2}\big)\big). \qquad (22)$$

The parameter s may be considered as spin in the auxiliary space and therefore may take integer or half-integer values, but generally speaking the right-hand side in (22) is well defined for arbitrary complex s.

From this definition immediately follows the equation

$$T_{-s-1}(\lambda) = -T_s(\lambda) \qquad (23)$$

and, for particular values of s, we have

$$T_{1/2}(\lambda) = T(\lambda) \qquad T_{-1/2}(\lambda) = 0 \qquad T_{-1}(\lambda) = -T_0(\lambda) = -\lambda^N. \qquad (24)$$

For the sake of brevity we shall use the following notation:

$$\Delta(a, b) \equiv P(a)\, Q(b) - P(b)\, Q(a). \qquad (25)$$

The function $\Delta(a, b)$ changes sign while $a \to b$, which implies the identity

$$\Delta(a, b)\, Q(c) + \Delta(b, c)\, Q(a) + \Delta(c, a)\, Q(b) = 0. \qquad (26)$$

Making use of definitions (22) and (25) we can rewrite the last equation as follows:

$$\begin{aligned} &T_{s_1}(\lambda + \mathrm{i}(s_2 - s_3)/3)\, Q(\lambda + 2\mathrm{i}(s_3 - s_2)/3) + T_{s_2}(\lambda + \mathrm{i}(s_3 - s_1)/3)\, Q(\lambda + 2\mathrm{i}(s_1 - s_3)/3) \\ &\quad + T_{s_3}(\lambda + \mathrm{i}(s_1 - s_2)/3)\, Q(\lambda + 2\mathrm{i}(s_2 - s_1)/3) = 0 \end{aligned} \qquad (27)$$

$s_1 + s_2 + s_3 + \frac{3}{2} = 0.$

Apparently this equation may be considered as a generalization of the $T - Q$ equation (5).

† See also [8].

Another simple identity

$$\Delta(a,b)\,\Delta(c,d) - \Delta(a,c)\,\Delta(b,d) + \Delta(a,d)\,\Delta(b,c) = 0 \tag{28}$$

leads to the following quadratic relations:

$$\begin{aligned} &T_{s_1}\big(\lambda - \mathrm{i}\big(s_1+\tfrac{1}{2}\big)\big)T_{s_3-s_2-1/2}(\lambda - \mathrm{i}(s_2+s_3+1)) \\ &\qquad -T_{s_2}\big(\lambda - \mathrm{i}\big(s_2+\tfrac{1}{2}\big)\big)T_{s_3-s_1-1/2}(\lambda - \mathrm{i}(s_1+s_3+1)) \\ &\qquad +T_{s_3}\big(\lambda - \mathrm{i}\big(s_3+\tfrac{1}{2}\big)\big)T_{s_2-s_1-1/2}(\lambda - \mathrm{i}(s_1+s_2+1)) = 0. \end{aligned} \tag{29}$$

For $s_2 = -1$, $s_3 = 0$, the last equation due to (23) and (24) may be written as the famous fusion relations:

$$T_s\big(\lambda - \mathrm{i}\big(s+\tfrac{1}{2}\big)\big)T(\lambda) = (\lambda + \mathrm{i}/2)^N T_{s-1/2}(\lambda - \mathrm{i}(s+1)) + (\lambda - \mathrm{i}/2)^N T_{s+1/2}(\lambda - \mathrm{i}s) \tag{30}$$

where $T_s(\lambda)$ is the eigenvalue of transfer matrix of quantum spin $\frac{1}{2}$ and auxiliary spin s.

3. XXX_{s_q}-model

Now let us consider the inverse situation where the quantum spin is s_q, while the auxiliary spin is $\frac{1}{2}$. This situation corresponds to the XXX_{s_q} spin chain. The above discussion could be easily generalized for this case.

Indeed, the Bethe ansatz equations have the following form (see, e.g. [2]):

$$\left(\frac{\lambda_j + \mathrm{i}\,s_q}{\lambda_j - \mathrm{i}\,s_q}\right)^N = \prod_{k\neq j}^{n} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k - \mathrm{i}} = -\prod_{k=1}^{n} \frac{\lambda_j - \lambda_k + \mathrm{i}}{\lambda_j - \lambda_k - \mathrm{i}} \qquad (j = 1, 2, \ldots, n) \tag{31}$$

where the notation is the same as in (1).

Now the set of meaningful solutions $\{\lambda_j\}$ are those for $n \leqslant s_q\,N$. The eigenvalues of the transfer matrix are given by

$$T_{1/2,s_q}(\lambda) = (\lambda + \mathrm{i}s_q)^N \prod_{j=1}^{n} \frac{\lambda - \lambda_j - \mathrm{i}}{\lambda - \lambda_j} + (\lambda - \mathrm{i}s_q)^N \prod_{j=1}^{n} \frac{\lambda - \lambda_j + \mathrm{i}}{\lambda - \lambda_j} \tag{32}$$

while the $T - Q$ Baxter equations look like

$$(\lambda - \mathrm{i}s_q)^N Q_{s_q}(\lambda + \mathrm{i}) + (\lambda + \mathrm{i}s_q)^N Q_{s_q}(\lambda - \mathrm{i}) = T_{\frac{1}{2}s_q}(\lambda; s_q)\, Q_{s_q}(\lambda). \tag{33}$$

To simplify further considerations we shall limit ourselves to the case $s_q = \frac{3}{2}$.

As in the case of $s_q = \frac{1}{2}$, we divide both sides of (33) by the product $Q(\lambda - \mathrm{i})\,Q(\lambda)\,Q(\lambda + \mathrm{i})$. However, in trying to represent the right-hand side as a finite difference, we meet an obstacle due to the different shifts of the spectral parameters in the numerators and denominators of the fractions. To overcome this difficulty we have to multiply both sides by the additional multipliers $(\lambda + \mathrm{i}/2)^N(\lambda - \mathrm{i}/2)^N$ (in the general case the number of these auxiliary multipliers is $2s_q - 1$):

$$\frac{T(\lambda)(\lambda + \mathrm{i}/2)^N(\lambda - \mathrm{i}/2)^N}{Q(\lambda + \mathrm{i})\,Q(\lambda - \mathrm{i})} = R(\lambda - \mathrm{i}/2) + R(\lambda + \mathrm{i}/2) \tag{34}$$

where

$$R(\lambda) = \frac{(\lambda - \mathrm{i})^N \lambda^N (\lambda + \mathrm{i})^N}{Q(\lambda - \mathrm{i}/2)\,Q(\lambda + \mathrm{i}/2)}. \tag{35}$$

Table 2. The illustrative example for the case $s_q = \frac{3}{2}$, $N = 2$.

Number	S	$Q(\lambda)$	$(2S+1)\,i\,P(\lambda)$	$T(\lambda)$
1	0	$\lambda^3 + \frac{5}{4}\lambda$	$\lambda^4 + \frac{5}{2}\lambda^2 + \frac{9}{16} + \alpha(\lambda^3 + \frac{5}{4}\lambda)$	$2\lambda^2 + \frac{15}{2}$
2	1	$\lambda^2 + \frac{9}{20}$	$\lambda^5 + \frac{5}{2}\lambda^3 + \frac{9}{16}\lambda + \alpha(\lambda^2 + \frac{9}{20})$	$2\lambda^2 + \frac{11}{2}$
3	2	λ	$\lambda^6 + \frac{15}{4}\lambda^4 + \frac{59}{16}\lambda^2 + \frac{45}{64} + \alpha\lambda$	$2\lambda^2 + \frac{3}{2}$
4	3	1	$\lambda^7 + \frac{91}{20}\lambda^5 + \frac{91}{16}\lambda^3 + \frac{369}{320}\lambda + \alpha$	$2\lambda^2 - \frac{9}{2}$

Further steps are the same as above and finally we arrive at the following fundamental relation:

$$P_{3/2}(\lambda + \mathrm{i}/2)Q_{3/2}(\lambda - \mathrm{i}/2) - P_{3/2}(\lambda - \mathrm{i}/2)Q_{3/2}(\lambda + \mathrm{i}/2) = (\lambda - \mathrm{i})^N \lambda^N (\lambda + \mathrm{i})^N \tag{36}$$

and expression for eigenvalues of transfer matrix $T_{1/2\,3/2}(\lambda)$:

$$T_{1/2\,3/2}(\lambda)(\lambda + \mathrm{i}/2)^N(\lambda - \mathrm{i}/2)^N = P_{3/2}(\lambda + \mathrm{i})\,Q_{3/2}(\lambda - \mathrm{i}) - P_{3/2}(\lambda - \mathrm{i})\,Q_{3/2}(\lambda + \mathrm{i}). \tag{37}$$

In conclusion of this section we formulate the second theorem, which generalizes the first one.

Theorem 2. *For each solution of equations (31) with $n \leqslant sN$, there exists a one-parametric associated solution with $n^* = 2sN - n + 1 > sN$.*

Note that with a fundamental relation of the type (36) for arbitrary s_q we can obtain the rational analogues of all fusion relations considered in [10].

4. Trigonometric case: XXZ spin chain

Here we shall consider the Bethe ansatz 'beyond the equator' for the XXZ spin chain. The general ideas of this generalization are the same as in the first section.

We shall use Baxter's parametrization (see, e.g., [6]) for spectral ϕ and crossing η parameters. In this notation the $T - Q$ Baxter equation looks like

$$T(\phi)\,Q(\phi) = \sin^N(\phi + \eta)\,Q(\phi - 2\eta) + \sin^N(\phi - \eta)\,Q(\phi + 2\eta). \tag{38}$$

As usual, the q-parameter of the XXZ model is defined by $q = \mathrm{e}^{2i\eta}$.

Recall that

$$Q(\phi) = \prod_{j=1}^{n} \sin(\phi - \phi_j) \tag{39}$$

is now a trigonometric polynomial of degree $n \leqslant N/2$, where a set $\{\phi_j\}$ substitutes the set of $\{\lambda_j\}$ in (1), all other notation was introduced in the first section.

Eigenvalues of the transfer matrix $T(\phi)$ are also trigonometric polynomials of degree N. Instead of the rational function $R(\lambda)$ we now have the meromorphic function

$$R(\phi) = \frac{\sin^N \phi}{Q(\phi - \eta)\,Q(\phi + \eta)}. \tag{40}$$

The analogue of the decomposition on the primitive fractions in the trigonometric case is the decomposition of (40) on to the primitive functions $1/\sin(\phi - \phi_j \pm \eta)$ for odd N and $\cot(\phi - \phi_j \pm \eta)$ for even N.

Making use of this expansion, and taking into account the Bethe ansatz, we obtain the trigonometric analogue of representation (12):

$$R(\phi) = \pi(\phi) + \frac{q(\phi+\eta)}{Q(\phi+\eta)} - \frac{q(\phi-\eta)}{Q(\phi-\eta)} \tag{41}$$

where $\pi(\phi)$ is the trigonometric polynomial of degree $N-2n$, while $\deg q(\phi) < n$.

Now the construction of the $P(\phi)$ which is the analogue of $P(\lambda)$ is reduced to the construction of the trigonometric polynomial $\rho(\phi)$ satisfying

$$\rho(\phi+\eta) - \rho(\phi-\eta) \equiv \pi(\phi). \tag{42}$$

In the present paper we shall consider the case where the q parameter is not the root of unity, i.e. η is not the rational part of π.

In this case $\sin(k\eta) \neq 0$, $k \in \mathbb{Z}$ and so we can use the following simple formulae:

$$\begin{aligned} \sin(k\phi) &= \frac{\cos(k(\phi-\eta)) - \cos(k(\phi+\eta))}{2\sin(k\eta)} \\ \cos(k\phi) &= \frac{\sin(k(\phi+\eta)) - \sin(k(\phi-\eta))}{2\sin(k\eta)}. \end{aligned} \tag{43}$$

For odd N, the degree of $\pi(\phi)$ is also odd and it may be decomposed into the harmonics $\cos(k\phi)$, $\sin(k\phi)$ $k \neq 0$.

In this case equations (43) solve the problem (42) and $\rho(\phi)$ is the trigonometric polynomial of the degree $N-2n$. The polynomial

$$P(\phi) \equiv \rho(\phi)\, Q(\phi) + q(\phi) \tag{44}$$

is the second solution of (38); its degree is $N-n$.

Apparently its decomposition

$$P(\phi) = \text{constant} \prod_{j=1}^{n^*} \sin(\phi - \phi_j^*) \tag{45}$$

where $n^* = N - n$ gives the solution for the trigonometric Bethe ansatz equation

$$\left(\frac{\sin(\phi_j+\eta)}{\sin(\phi_j-\eta)}\right)^N = \prod_{k\neq j}^{n^*} \frac{\sin(\phi_j-\phi_k+2\eta)}{\sin(\phi_j-\phi_k-2\eta)} \qquad (j = 1, 2, \ldots, n^*) \tag{46}$$

'beyond the equator'.

For even N, the polynomial $\pi(\phi)$ has the zero harmonic and therefore the solution of (42) acquires a term with a linear (nonperiodic) dependence on ϕ.

As the result we have the following.

Theorem 3 (the associated solution of the Baxter equation for a XXZ spin chain). *For each solution of equation (38) with $n \leqslant N/2$ there exists the associated solution, which is a trigonometric polynomial of degree $n^* = N - n > N/2$ in the case of odd length N.*

For even length the associated solution has the form (44) where $\rho(\phi)$ contains the linear (nonperiodic) dependence on ϕ.

For the construction of the fusion relation in the case of the XXZ model it is sufficient to use two main ingredients—the analogues of equations (21) and (22). The first one can be extracted from the representation for $R(\phi)$:

$$P(\phi+\eta)\, Q(\phi-\eta) - P(\phi-\eta)\, Q(\phi+\eta) = \sin^N \phi. \tag{47}$$

The second one may be written as follows:

$$T_s(\phi) = P(\phi+(2s+1)\eta)\, Q(\phi-(2s+1)\eta) - P(\phi-(2s+1)\eta)\, Q(\phi+(2s+1)\eta). \tag{48}$$

Therefore, all the results of sections 2 and 3 holds true for a generic XXZ spin chain.

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